

A tutorial on chaotic dynamics for economists

Hung T. Nguyen

*Department of Mathematical Sciences, New Mexico State University, Las Cruces,
New Mexico (USA)
& Faculty of Economics, Chiang Mai University, Chiang Mai (Thailand)*

Abstract

Purpose – Presenting a simplest tutorial for new comers to the field of chaotic dynamics in economics and finance.

Design/methodology/approach – To attract economists, the paper is “written in a way that doesn’t kill it”.

Findings – Elaborating in simplest terms the complex notion of chaotic dynamics.

Originality/value – Aiming at attracting economists to chaotic dynamical approach to economics and finance.

Keywords Apparent randomness, chaos, chaotic dynamics, endogeneous deterministic dynamics, exogeneous random shocks, Lyapunov exponent, SDIC.

Paper type General tutorial

1. Introduction

This tutorial is written for economists who possibly heard about chaotic dynamics and wish to know what are chaotic dynamics, why chaotic dynamics are useful in economics, and how to apply chaotic dynamics in economic problems. Since the foundation of chaotic dynamics is heavily theoretical, the writing of a tutorial on it, specifically to attract the attention of new comers to the field, is a challenge! The tutorial should attract economists rather than putting them away! As such, I will follow the advice of my late friend, Professor Elbert A. Walker: “Write it in a way that doesn’t kill it”.

Statisticians and economists, in particular, are familiar with learning from data (either by Statistics, e.g., Spiegelhalter, 2020, or by Machine Learning, e.g., Ananthaswamy, 2025), i.e., looking for patterns in observed data of phenomena, since it is the goal of prediction in interested problems, as it provides “useful” models for investigations.

So far, we are quite familiar with traditional methods for considering models for prediction such as (linear) stochastic time series models (e.g., for observed time-dependent data), i.e., models explaining, say, the relationship (patem) of a variable in terms of the covariates $X_1, X_2, \dots, X_k$ as an ex-pression of the form

$$Y = \varphi(X) + \varepsilon$$

where ε is a “noise” (an exogeneous random shock!), $X = (X_1, X_2, \dots, X_k)$, and $\varphi(X_1, X_2, \dots, X_k)$ is linear in the sense that it depends only on each X_j , and not on any raised to some power, or on any product of the X_j s, i.e.,

$$\varphi(X_1, X_2, \dots, X_k) = \sum_{j=1}^k a_j X_j$$

Learning the weights (or parameters) a_j consists of figuring out algorithms to identify them, either by statistical estimation or by machine learning methods.

Clearly, linear models are too restricted as they might not be able to capture the “true” relationship between Y and X , given the data.

Current efforts in the literature are focusing on nonlinear models, especially for time series data, such as nonlinear autoregressive models. It is precisely here that something unexpected arises!

Remark. Dynamical systems are about things that change in time (continuous or discrete). A map (or model) is a function $\varphi(\cdot): X \rightarrow X$ (from some “phase” space to itself). In discrete time, the dynamic of $\varphi(\cdot)$ is given by $x_{n+1} = \varphi(x_n)$, and an orbit (or trajectory), starting at some initial condition x_o , is $\{\varphi^n(x_o), n = 0, 1, 2, \dots\}$ by iterations, where $\varphi^n(x_o)$ denotes $\varphi \circ \varphi \circ \dots \circ \varphi(x_o)$, n times. We are interested in the long term behavior of the dynamic for prediction purpose.

When the (deterministic) part $\varphi(\cdot)$ is nonlinear, its dynamic can be chaotic causing by itself (i.e., without the exogeneous random shock ε) the unpredictability, in long term. This is so since the deterministic part of a stochastic time series model (with additive or multiplicative noise) produces apparent randomness that, in turn, causes the unpredictability of the model in a long run.

Recall that we consider the problem of prediction because of the uncertainty in future values. If the model produces unpredictability then prediction is impossible! Thus, to handle the apparent randomness (when the deterministic dynamics of the model is chaotic) we need, first of all, to “figure out” the real causes of the fluctuations of our observed data, namely the fluctuations of our data are due to exogeneous random shocks (like assumed in EMH), or to the endogenous chaotic dynamics of the deterministic part of our model, or both (in a hybrid model)?

Following such analyses, we should be able to address our prediction problem with confidence!

This tutorial aims simply at spelling out what we need to know when facing chaotic dynamics in our data. Specifically, the rest of this tutorial consists of answering three basic questions, in this order,

What? Why? and How?

2. What are chaotic dynamics?

What are chaotic dynamics? Upfront: They are deterministic dynamics exhibiting “sensitive dependence on initial conditions” (SDIC) that, in turn, create apparent randomness.

As such, their long term behaviors are unpredictable. In a tutorial such as this, it suffices to elaborate on SDIC and apparent randomness.

A standard time series stochastic model with additive noise, in discrete time, is of the form

$$x_{n+1} = \varphi(x_n) + \varepsilon_{n+1}$$

where the deterministic map $\varphi(\cdot)$ is a map on some phase (state) space, e.g., .

Suppose we ignore the random noise ε , and look at the relation $x_{n+1} = \varphi(x_n)$ only to find out whether, by itself, the deterministic “model” $\varphi(\cdot)$ could contribute to the fluctuations of our observed time series data.

If $\varphi(\cdot)$ is linear, such as $\varphi(x) = \alpha x$, then the behavior of this relation $\varphi(\cdot)$ is predictable, so that the unpredictability of our observed time series data must be due to some exogeneous random shocks!

Indeed, for the linear dynamic $\varphi(x) = \alpha x$, its orbit (trajectory, obtained by iterations), starting at some initial value $x_o > 0$, is $\{\alpha^n x_o, n \geq 1\}$, and hence is predictable. Specifically, if $\alpha = 1$, then the associated orbit is constant; if $\alpha < 0$, then the orbit tends to zero, as $n \rightarrow \infty$; and if $\alpha > 0$, then the orbit tends to infinity as $n \rightarrow \infty$.

How about deterministic nonlinear maps (models)? such as

$$\varphi(x) = \sqrt{a + bx^2}$$

in a stochastic ARCH(1) model (with multiplicative noise)? What is the behavior of a nonlinear deterministic dynamic? Are there special ones that contribute, by themselves, to the unpredictability of the time series observations? If there are such dynamics, they will be called chaotic dynamics!

In population growth modeling, it is more realistic to consider models of the form

$$y_{n+1} = \psi(y_n) = \alpha y_n - \beta y_n^2$$

i.e., with deterministic dynamic given by the nonlinear map

$$\psi(y) = \alpha y - \beta y^2$$

But if we let $x = \left(\frac{\beta}{\alpha}\right)y$, then we have

$$x_{n+1} = \alpha x_n (1 - x_n)$$

i.e. the model

$$\varphi(x) = \alpha x(1-x)$$

It turns out (see Section 4) that the behavior of the dynamic of this logistic map when $\alpha = 4$ is the same as that of a much simpler model (that we can investigate without a computer !), namely the doubling map on $[0,1]$:

$$\varphi(x) = 2x \pmod{1}$$

where (mod 1/ or modulo 1) means removing the integer part of $2x$, and keeping its fractional part. For example, if $x = 0.70$, then $(2x) \pmod{1}$ is 0.40.

As we will see, the deterministic dynamic of this map is sensitive dependence on initial conditions, and by itself, creates apparent randomness, resulting in unpredictability of its behavior.

Recall that in discrete time, an orbit (trajectory) of this dynamical system is obtained by iterations from some initial position $x_o \in [0,1]$, i.e., $\{\varphi^n(x_o), n \geq 1\}$, here $\varphi^n(x_o) = 2^n x_o \pmod{1}$.

For a short time like $n = 10$, we have, for $x_o = 0.40$, $\varphi^n(x_o) = 2^n x_o = 0.60$, whereas for a very nearby initial state of 0.40, say, $x_o = 0.41$, we have $\varphi^{10}(0.41) = 2^{10}(0.41) = 0.86$.

We realize the sensitivity to initial conditions of the dynamic of this simple map $\varphi(x) = 2x \pmod{1}$? A very small change in the initial condition value (an almost insignificant change) led to a completely different outcome.

Specifically, the SDIC of the dynamic of $\varphi(\cdot)$ is formulated as follows. For any $x \in [0,1]$, if there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exist $y \in [0,1]$ and $n \geq 1$ such that $|y - x| < \delta$ and $|\varphi^n(y) - \varphi^n(x)| > \varepsilon$. The above doubling dynamics does have SDIC. It is not easy to use its iterates to show its SDIC ! A technique called Symbolic Dynamics will be used to achieve the result (See Section 4); in other words, the apparent randomness of the dynamics $\varphi(\cdot)$ will be explained by using symbolic dynamics. Also, as probability quantifies randomness, the so-called Lyapunov exponent quantifies SDIC (more in Section 4).

Remark. The dynamic of a map $\varphi(\cdot)$ is chaotic if it has the following properties (to be explained in full details in Section 4):

- (i) It is SDIC,
- (ii) It has a dense set of periodic orbits,
- (iii) It has at least one dense orbit.

For a survey of chaotic dynamics in economics and finance, see e.g., Guegan (2009). For an introduction to chaos theory with complete mathematical background, see the references Bishop (2025), Doyné Farmer (2024), Feldman (2019), and Johnson, Jefferies and Hui (2003).

What is the effect of SDIC?

The effect is that, since in practice, we never know the true initial condition of a dynamical system, so that long-term prediction, even with deterministic dynamics, becomes practically impossible. In other words, such dynamics create unpredictability like randomness (apparent randomness).

As stated in the introduction, and given the fact that chaotic dynamics theory is highly theoretical with “sophisticated” mathematics, for “not killing the topic for beginners”, this Section will provide simply the explanation of “What is a chaotic dynamic?”, namely, it is a deterministic dynamic that at least exhibits the sensitive dependence on initial conditions.

As the readers get through Sections 2 and 3, a summary of a complete definition of chaotic dynamics will be given in Section 4.

For now, let’s spell out what is “apparent randomness”? In other words, how to relate chaos to randomness?

Well, we are interested in prediction when facing uncertainty. Uncertainty can appear in two different reasons. Either we face (true) randomness, or lack of knowledge (epistemic uncertainty), such as not being able to know the true initial position of a chaotic dynamic.

Remark. These two types of uncertainty create frequentist and Bayesian schools of thought !

Thus, roughly speaking, the SDIC of a chaotic dynamic is the cause of epistemic uncertainty about the behavior of the dynamic, i.e., making its predictability impossible, similar to true randomness, termed “apparent randomness”. The divergence of nearby orbits is the reason why chaos leads to unpredictability.

In summary, chaos theory is the study of deterministic dynamical systems that appear random and unpredictable.

3. Why chaotic dynamics are useful for economics?

Upfront: An answer could be found in the book “Making Sense of Chaos: A Better Economics for a Better World” by J. Doyné Farmer (2024). See also Guegan (2009).

Economic and financial issues are complex issues. Without “economic laws”, we rely upon economic models. As in statistics where “all models are wrong, but some are useful”, we could try to specify a bit of “useful models”. Put it differently, we could ask “What makes a model useful?”. Obviously, we are talking about modeling of, say, the dynamics of time series observed data.

Traditional time series models are stochastic linear models, in the sense that their deterministic part are linear functions, and the models are stochastic since we add to them an exogenous random shock (noise), additively or multiplicatively, to explain their randomness behavior.

However, linear dynamics are very special cases of models for real economic and financial problems. In fact, static models, such as models of population growth, models for commodity prices, models for advertising expenditures or cobweb models, can be nonlinear.

As pointed out in the previous Section, when the deterministic dynamic of a model (static or stochastic) is nonlinear, it can be chaotic so that the causes of random fluctuations from the model should be investigated with great care to reflex its reasonable behavior, say, for prediction.

It all boils down to the modeling problem from observed data. The knowledge of chaotic dynamics in nonlinear models in finance, in particular, should ring the bell for financial modeling.

There is no free lunch in finance? This is what the Efficient Market Hypothesis (EMH) claims, namely asset prices follow a random walk (in price space). Specifically, fluctuations of time series data are due solely to exogenous random shocks. This might be possible if we model asset prices by stochastic linear models. More realistic nonlinear models might deviate from the EMH. This is so since, as a challenge to EMH, market deterministic dynamics, without exogenous random shocks, can be unpredictable when they are SDIC.

Research in the literature reveals that economists are interested in chaotic dynamics because it offers a framework for understanding better the complex fluctuations of economics, such as business cycles, markets crashes. Technically speaking, the contribution of chaotic dynamics is mainly based on the fact that it allows for endogenous explanation of apparent randomness of fluctuations, without relying solely on exogenous random shocks. “Useful models” could be hybrid in which, the causes of uncertainty and unpredictability could be both exogenous and endogenous.

In summary, just like in physics, chaotic dynamics are useful for economics and finance when their models are nonlinear and exhibit sensitive dependence on initial conditions.

4. How to apply chaotic dynamics?

To answer this question for “empirical researchers”, it is necessary to evoke some “elegant” mathematics, somewhat similar to the writing of Ananthaswamy (2025) in Machine Learning, to attract them. It is possible to do that without “killing” the chaos theory, i.e., “repelling” empirical researchers, as Elbert Walker once reminded us. Of course, for “theoretical researchers”, the references at the end of this paper are recommended. In other words, in the language of dynamical systems, we wish to make chaos an attractor rather than a repeller to new comers to the field!

In Section 2 we gave a simple example of a deterministic nonlinear dynamic that we call a chaotic dynamic that is sensitive dependence on initial conditions (SDIC), and hence exhibiting apparent randomness. As such, chaotic dynamics in time series models could be the endogenous cause of their unpredictability, even in the absence of exogenous random shocks.

To appreciate the usefulness of chaotic dynamics for modeling in, say, economics, let’s take a closer look at the doubling map on $[0,1]$, namely

$$\varphi(\cdot): [0,1] \rightarrow [0,1]; \varphi(x) = 2x \pmod{1}$$

We proceed now to verify that the dynamic of the doubling map $\varphi(\cdot)$ is chaotic, i.e., it has the following properties (of its behavior):

- (i) There exists a point $x_0 \in [0,1]$ such that the orbit $\{\varphi^n(x_0), n \geq 0\}$ is dense in $[0,1]$,
- (ii) Sensitive dependence on initial conditions,
- (iii) The set of periodic points of the iterated of $\varphi(\cdot)$ is dense in $[0,1]$.

Remark. A (proper) subset A of $[0,1]$ is dense if for any $x \in [0,1]$, and any $\delta > 0$, there is $y \in A$ such that $|x - y| < \delta$.

First, here is a clever way (“elegant” mathematics, namely the so-called Symbolic Dynamics) to establish the property (iii) for the doubling map.

Since each $x \in [0,1]$ can be represented by its binary expansion, the map $\varphi(\cdot)$ above induces a Shift map $S(\cdot)$ on the space Ω of binary expansions of elements of $[0,1]$.

Specifically, each $x \in [0,1]$ has a binary expansion $x = \sum_{j=1}^{\infty} \frac{\omega_j}{2^{j+1}}$, with $\omega_j \in \{0,1\}$, so that $\varphi(x) = \varphi(\omega_0, \omega_1, \omega_2, \omega_3, \dots) = 2x \pmod{1} = (\omega_1, \omega_2, \omega_3, \dots)$ i.e., $\varphi(\cdot)$ induces a “left shift” map on Ω :

$$S(\omega_0, \omega_1, \omega_2, \omega_3, \dots) = (\omega_1, \omega_2, \omega_3, \dots)$$

from it we can deduce important properties of $\varphi(\cdot)$ by looking at $S(\cdot)$ on Ω .

Without going into details, for example, using $S(\cdot)$, we can show that the set of all periodic points of the iterated doubling map is dense in $[0,1]$ which is the property (iii) in the above definition of chaotic dynamics. Note that all such period points are unstable and hence the dynamics will in fact behave aperiodically.

It turns out that using binary expansions of numbers $x \in [0,1]$, the SDIC (property (ii)) of the doubling map can be established.

For the property (i), we can use again the same clever technique based on binary expansions of $x \in [0,1]$ to construct an x_0 such that the orbit $\{\varphi^n(x_0) : n \geq 0\}$ of $\varphi(\cdot)$ at x_0 is dense in $[0,1]$.

In summary, using the Shift map $S(\cdot)$, the dynamics of the doubling map (i.e., $x_{n+1} = \varphi(x_n)$) can be shown to be a chaotic.

Remark. It turns out that if $\varphi(\cdot)$ is chaotic then any dynamics conjugated to it are also chaotic, where two dynamics $f(\cdot) : X \rightarrow X$, and $g(\cdot) : Y \rightarrow Y$ (on topological spaces, e.g., intervals of the real line) are said to be conjugated if there exists a “homomorphism” $h(\cdot) : X \rightarrow Y$, such that $f \circ h = h \circ g$. When $f(\cdot)$, and $g(\cdot)$ are conjugated, they have the same dynamic behavior. Conjugacy of two dynamics happens when they are related by a change of coordinates.

A function $h(\cdot) : X \rightarrow Y$ is a homomorphism if it is a bijection and both h and h^{-1} are continuous. For example, $h(x) = \sin^2\left(\frac{\pi}{2}x\right)$ on $X = [0,1] = Y$; $h(x) = x^n$ on $[-1,1] = X = Y$.

With appropriate changes of coordinates, it can be checked that the logistic and the Tent maps are conjugated. Specifically, which the change of coordinates $x = \sin^2\left(\frac{\pi}{2}t\right)$, the logistic map $L(x) = 4x(1-x)$ and the the Tent map $T(\cdot)$ are conjugated, where

$$T(t) = \begin{cases} 2t & \text{for } t \in [0, 0.5] \\ 2(1-t) & \text{for } t \in [0.5, 1] \end{cases}$$

In applications, how to detect that a dynamic is chaotic? A measure of “chaos” is the Lyapunov exponent. What is the Lyapunov exponent?

The simplest way to arrive at a standard method to detect chaos of a dynamics (finding a quantitative measure of chaos, in the sense of quantitatively assessing the rate of expansion/contraction at a point produced by the iteration of a map/ dynamic) is considering a simplest (specific) example: the above Tent

Consider two nearby points (states) x and $y = x + \varepsilon$, with $|\varepsilon|$ very small (in the same side of the point $\frac{1}{2}$, noting that $T(\cdot)$ is differentiable on $[0,1]$ except at $\frac{1}{2}$, with $|T'(x)| = 2$).

The separation between $T(x)$ and $T(y)$ is

$$|T(x) - T(y)| = |T(x) - T(x + \varepsilon)| = 2|x - y| = 2|\varepsilon|$$

and the separation between $x_n = T^n(x)$ and $y_n = T^n(y)$, for $n \geq 0$, is

$$|T^n(x) - T^n(y)| = |T^n(x) - T^n(x + \varepsilon)| = 2^n |\varepsilon|$$

And

$$\left| \frac{T^n(x) - T^n(x + \varepsilon)}{\varepsilon} \right| = 2^n$$

As $\varepsilon \rightarrow 0$, we arrive at

$$|(T^n)'(x)| = 2^n = (e^{\log 2})^n = e^{n \log 2}$$

The number $\lambda(x) = \lambda = \log(2)$ (constant on $[0,1]$) is called the Lyapunov exponent of the Tent map.

The Lyapunov exponent measures the (infinitesimal) exponential rate at which nearby orbits are moving away. A positive Lyapunov exponent is an operational definition of chaotic behavior of nonlinear dynamics (Thus, this Tent map is chaotic).

A formula for Lyapunov exponent in general is derived as follows.

Let $\varphi(\cdot): X \rightarrow X$ be a differentiable map (for X an interval of R).

Like for the Tent map, we have

$$|\varphi^n(x) - \varphi^n(x + \varepsilon)| \approx e^{n\lambda(x)} |\varepsilon|$$

or

$$\left| \frac{\varphi^n(x) - \varphi^n(x + \varepsilon)}{\varepsilon} \right| \approx e^{n\lambda(x)}$$

Take the limit as $\varepsilon \rightarrow 0$, we have

$$e^{n\lambda(x)} \approx \lim_{\varepsilon \rightarrow 0} \left| \frac{\varphi^n(x) - \varphi^n(x + \varepsilon)}{\varepsilon} \right| = \left| \lim_{\varepsilon \rightarrow 0} \frac{\varphi^n(x) - \varphi^n(x + \varepsilon)}{\varepsilon} \right| = |(\varphi^n)'(x)|$$

Taking logarithms, we get

$$\lambda(x) \approx \frac{1}{n} \log \left[|(\varphi^n)'(x)| \right]$$

Thus, we have the definition of Lyapunov exponent of $\varphi(\cdot)$ at $x \in X$:

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[|(\varphi^n)'(x)| \right]$$

If $\lambda(x)$ is independent of x on some subset A of X , i.e., $\lambda(x) = \lambda$, a constant, for all $x \in A$, then we say that the Lyapunov exponent λ for $\varphi(\cdot)$ over A .

For example, the Lyapunov exponent of the Tent map is $\lambda = \log(2) > 0$, so that its dynamic is chaotic.

The Lyapunov exponents of the logistic map are as follows.

$$\varphi_\theta(x) = \theta x(1-x)$$

with $x \in [0, 1], \theta \in (0, 4]$.

(1) For $0 < \theta \leq 1$: We have $\lambda(x) = \lambda \log |\varphi'(0)| = \log \theta < 0$ (0 is the unique stable fixed point).

(2) For $1 < \theta \leq 3$: Stable fixed point is $1 - \frac{1}{\theta}$.

$$\lambda = \lambda(x) = \log \left| (L_\theta)' \left(1 - \frac{1}{\theta} \right) \right| = \log |2 - \theta| < 0$$

(3) $3 < \theta < 1 + \sqrt{6}$: Attractive 2-cycle $\{p, q\}$:

$$\lambda(x) = \frac{1}{2} \log \left| L'_\theta(p) L'_\theta(q) \right| < 0$$

(4) $\theta = 4$: $\lambda = \log(2) > 0$. The dynamic of the Ulam map $L(x) = 4x(1-x)$ is chaotic.

Remark. The notion of Lyapunov exponent can be discussed in a general framework. Here, we just mention it in the context of one-dimensional maps. Lyapunov exponents measure the (infinitesimal) exponential rate at which nearby orbits are moving away. A positive Lyapunov exponent is an operational definition of chaotic behavior of nonlinear dynamics.

Suppose we come up with a deterministic (nonlinear) dynamics $\varphi(\cdot)$ for our observed time series data (for a model). Before “go ahead” to add or multiply it by a random error term (to use traditional stochastic approach), should we ask whether $\varphi(\cdot)$ is chaotic or not? That requires a “measure of chaos” to decide.

So, how to quantify (or measure) chaos to decide that a deterministic dynamics is chaotic or not?

Now, as chaos is essentially characterized by the “sensitive dependence on initial conditions” (SDIC for short) we could explore this property to suggest a measure of chaos. Note that chaotic behavior of a dynamic is characterized by the exponential divergence of nearby orbits (trajectories).

The SDIC can be measured by the extent to which how the deviation of states from nearby initial conditions x_o and $x_o + \varepsilon$ behaves, as a function of “time” n .

Specifically, how the distance $|\varphi^n(x_o + \varepsilon) - \varphi^n(x_o)|$ behaves for large n ?

It is natural to look at $\frac{d\varphi^n(x_o)}{dx_o}$, as a measure of change of x_o . Now, for chaotic dynamics, separations of states between nearby initial conditions are very quick, in fact exponentially in time (as verified by computer iterations). As such, the approximation $dx_n = e^{\lambda n} dx_o$ suggests that small changes in initial conditions result in separation of paths at an exponential rate as n grows.

More specifically, from

$$|\varphi^n(x_o + \varepsilon) - \varphi^n(x_o)| \approx \varepsilon e^{\lambda n}$$

we get

$$e^{\lambda n} \approx \frac{|\varphi^n(x_o + \varepsilon) - \varphi^n(x_o)|}{\varepsilon}$$

so that

$$\lambda = \lim_{n \rightarrow \infty, \varepsilon \rightarrow 0} \left[\frac{1}{n} \log \left(\frac{|\varphi^n(x_o + \varepsilon) - \varphi^n(x_o)|}{\varepsilon} \right) \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \log \left(\left. \frac{d\varphi^n}{dx} \right|_{x=x_o} \right) \right]$$

Applying chain rule derivation, we obtain

$$\lambda = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{j=0}^{n-1} \log \left(\left. \frac{d\varphi}{dx} \right|_{x=x_j} \right) \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{j=0}^{n-1} \log(|\varphi'(x_j)|) \right]$$

The quantity λ is called the Lyapunov exponent that can be easy to compute numerically (for n large). It is a measure of chaos in the sense that positive $\lambda > 0$ means the dynamics is chaotic (since then nearby trajectories $\{\varphi^n(x_o), n \geq 0\}$, and $\{\varphi^n(x_o + \varepsilon), n \geq 0\}$ diverge exponentially, as time n grows).

Note that if $\lambda < 0$, the two trajectories converge and the system is not chaotic. If $\lambda = 0$, the trajectories diverge algebraically (i.e., as a power of n).

Such systems are said to exhibit weak chaos (whereas for $\lambda > 0$, the chaos is strong). Weak chaos do not have exponential divergence.

Example. Consider the Tent map with parameter:

$$\varphi_\theta(x) = \begin{cases} \theta x & \text{for } 0 \leq x \leq \frac{1}{2} \\ \theta - \theta x & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

on $[0,1]$ with parameter $\theta \in [0,2]$.

Let's find out for which values of θ , the dynamics is considered as chaotic. Since $\varphi'(x) = \pm\theta$ for all x , we get

$$\lambda = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{j=0}^{n-1} \log |\varphi'(x_j)| \right] = \lim_{n \rightarrow \infty} \left[\frac{\log \theta}{n} \sum_{j=0}^{n-1} (1) \right] = \log \theta$$

Thus, the tent map is chaotic for $\theta > 1$ since $\lambda = \log \theta > 0$.

In general you need a computer to calculate λ , such as for the logistic map !

Remarks.

(i) The above formula is obtained by chain rule derivation, as $\varphi^n(x_o) = \varphi(\varphi(\dots\varphi(x_o)))$. For example :

$$(\varphi^2)'(x_o) = (\varphi(\varphi(x_o)))' = \varphi'(\varphi(x_o))\varphi'(x_o) = \varphi'(x_1)\varphi'(x_o)$$

(ii) The fact that λ is independent of x_o can be justified by observing that the λ_j can be understood as the average of separations over the entire “strange attractor” as trajectories go almost everywhere on the attractor (statisticians can evoke the ergodic theorem or the law of large numbers to justify it). Lyapunov exponents measure the average stability properties of a dynamics on the attractor.

(iii) For a stable fixed point attractor of a two dimensional linear dynamical system, the Lyapunov numbers are the absolute values of the eigenvalues of the Jacobian matrix evaluated at the fixed point. Thus, Lyapunov exponents are generalizations of the familiar notion of eigenvalues.

(iv) The Lyapunov exponent of the Ulam map $\varphi(x) = 4x(1-x)$ is $\log(2)$, and hence the Ulam dynamics is chaotic.

Recall that the orbit of a map $\varphi(\cdot)$ starting at a point x , the Lyapunov exponent is

$$\lambda(\varphi, x) = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \log |(\varphi^n)'(x)| \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{j=0}^{n-1} \log |\varphi'(x_j)| \right]$$

For example, for the logistic map $\varphi_\theta(x) = \theta x(1-x)$, we compute λ by taking a large n from

$$\frac{1}{n} \left[\log |\theta(1-2x_{n-1})| + \log |\theta(1-2x_{n-2})| + \dots + \log |\theta(1-x_o)| \right]$$

If x and y are close, then $|\varphi(x) - \varphi(y)| \approx |\varphi'(x)| |x - y|$ because of Taylor formula assuring that $\varphi(y) = \varphi(x) + \varphi'(x)(y-x)$ plus something of the order $(y-x)^2$.

For all but a countable set of initial conditions x_o , the Lyapunov exponent of the Ulam map with initial condition x_o is $\log(2)$.

5. Conclusion

As a simplest tutorial to attract economists to the current literature on chaotic dynamics in economics and finance, this paper is designed to spell out, in simplest terms, the concept of chaotic dynamics. Specifically, we elaborate, without heavy mathematics, upon basic features of chaos, especially its sensitive dependence on initial conditions that creates unpredictability in long term via the apparent randomness. With such behavior of deterministic dynamics in mind, a closer look at the problem of modeling in applications seems desirable. Specifically, what are the causes of fluctuations of observed data? exogenous random shocks or endogeneous chaos? or both?

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Corresponding author

Hung T. Nguyen can be contacted at: hunguyen@nmsu.edu